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Effect Algebras and Tensor Products of S-Sets

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An S-set is an algebraic structure that generalizes an effect algebra. Unlike effect algebras, the tensor product of two S-sets always exists and this tensor product can be concretely represented. Morphisms are used to study relationships between S-sets and effect algebras. The S-set tensor product is employed to obtain information about effect algebra tensor products.

1. INTRODUCTION

Effect algebras (or D-posets) have been recently introduced as an axiomatic model for the foundations of quantum mechanics (Dvurečenskij and Pulmannová, 1994; Dvurečenskij and Riečan, 1994; Foulis and Bennett, 1994; Kôpka, 1982; Kôpka and Chovanec, 1994; Navara and Pták, n.d.). Tensor products of effect algebras are important because they are used to describe coupled physical systems (Aerts and Daubechies, 1975; Foulis, 1989; Foulis and Randall, 1980; Zecca, 1978), and various investigations concerning the existence of tensor products have been conducted (Dvurečenskij, 1995; Foulis and Bennett, 1993; Pulmannová, 1985). However, it has now been shown that the tensor product of two effect algebras need not exist (Gudder and Greechie, 1996). This suggests the following question: Is there a suitable generalization of an effect algebra for which tensor products always exist?

To answer this question, we introduce a structure called an S-set. An S-set is a generalization of an effect algebra and the tensor product of two S-sets always exists. Unlike previous effect-algebra tensor-product existence proofs, our proof gives a concrete representation for the tensor product of two S-sets. We also use morphisms to study relationships between S-sets and

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effect algebras. Finally, we employ S-set tensor products to obtain information about effect algebra tensor products.

2. DEFINITIONS

A *partial binary operation* on a nonempty set P is a map $\oplus$: $D(\oplus) \to P$ with domain $D(\oplus) \subseteq P \times P$. For simplicity, we write $a \oplus b$ for $\oplus\,(a, b)$. An S-*set* is an algebraic system $\mathcal{P} = (P, 0, 1, \oplus)$, where 0, 1 are distinct elements of P and $\oplus$ is a partial binary operation on P that satisfies the following conditions:

(S1) If $(a, b) \in D(\oplus)$, then $(b, a) \in D(\oplus)$ and $b \oplus a = a \oplus b$.
(S2) If $(a, b) \in D(\oplus)$ and $(a \oplus b, c) \in D(\oplus)$, then $(b, c) \in D(\oplus)$, $(a, b \oplus c) \in D(\oplus)$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(S3) For every $a \in P$, there exists a $b \in P$ such that $(a, b) \in D(\oplus)$ and $a \oplus b = 1$.
(S4) For every $a \in P$, $(0, a) \in D(\oplus)$ and $0 \oplus a = a$.
(S5) If $(a, b) \in D(\oplus)$ and $a \oplus b = 0$, then $a = b = 0$.

If $\oplus$ satisfies S1, S2, and S4, then $\mathcal{P}$ is a *partial Abelian monoid* (PAM) and if in addition $\oplus$ satisfies S5, then $\mathcal{P}$ is a *positive* PAM (Wilce, n.d.). Condition S3 ensures the existence of supplements. In particular, if $a \oplus b = 1$, we call b a *supplement* of a and we denote the set of supplements of a by $S(a)$. We denote the cardinality of a set A by $|A|$. An S-set that satisfies the following conditions is called an *effect algebra* (or D-*poset*) (Dvurečenskij and Pulmannová, 1994; Dvurečenskij and Riečan, 1994; Foulis and Bennett, 1994; Kôpka, 1982; Kôpka and Chovanec, 1994; Navara and Pták, n.d.):

(S6) If $(a, 1) \in D(\oplus)$, then $a = 0$.
(S7) $|S(a)| = 1$ for every $a \in P$.

It can be shown that S1–S3, S6, and S7 imply S4 and S5, so S4 and S5 are redundant for effect algebras (Dvurečenskij and Pulmannová, 1994; Foulis and Bennett, 1994; Navara and Pták, n.d.). We say that an S-set is *total* if $\oplus$ is a binary operation ($D(\oplus) = P \times P$). It follows from S6 that no nontrivial effect algebra is total.

Example 1. Let X be a nonempty set and let $P = 2^X$ be its power set. We define $0 = \varnothing$ and $1 = X$. Letting α be an infinite cardinal, define

$$D(\oplus) = \{(a, b) \in P \times P: \left|a \cap b\right| \le \alpha\}$$

and for $(a, b) \in D(\oplus)$, define $a \oplus b = a \cup b$. Then $\mathcal{P} = (P, 0, 1, \oplus)$ clearly satisfies S1 and S3–S5. To prove S2, suppose that $(a, b), (a \oplus b, c) \in D(\oplus)$. Then $|a \cap b| \le \alpha$ and

$$\left|(a \cap c) \cup (b \cap c)\right| = \left|(a \cup b) \cap c\right| \le \alpha$$

Hence, $|a \cap c|, |b \cap c| \leq \alpha$, and

$$|a \cap (b \cup c)| = |(a \cap b) \cup (a \cap c)| \leq |a \cap b| + |a \cap c| \leq 2\alpha = \alpha$$

Thus, $(b, c), (a, b \oplus c) \in D(\oplus)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. If $|X| \geq 2$, then $\mathcal{P}$ is an S-set that is not an effect algebra. If $|X| \leq \alpha$, then $\mathcal{P}$ is a total S-set and $\oplus = \cup$. If $|X| > \alpha$, then $\mathcal{P}$ is not total.

Example 2. Let $P = [0, 1] \subseteq \mathbb{R}$, define $0 = 0$, $1 = 1$, and $a \oplus b = \min\{a + b, 1\}$ for all $a, b \in P$. Then $\mathcal{P} = (P, 0, 1, \oplus)$ clearly satisfies S1 and S3–S5. To prove S2, if $a + b + c \leq 1$, then

$$(a \oplus b) \oplus c = a + b + c = a \oplus (b \oplus c)$$

and if $a + b + c > 1$, then $(a \oplus b) \oplus c = 1 = a \oplus (b \oplus c)$. Thus, $\mathcal{P}$ is a total S-set that is not an effect algebra.

Example 3. Let H be a complex Hilbert space and let P be the set of linear operators on H that satisfy $0 \leq A \leq I$ for all $A \in P$. Define $0 = 0$, $1 = I$, and, for $A, B \in P$, define

$$A \oplus B = \begin{cases} A + B & \text{if } A + B \in P \\ I & \text{if } A + B \notin P \end{cases}$$

Then, as in Example 2, it is easy to show that $\mathcal{P} = (P, 0, 1, \oplus)$ is a total S-set that is not an effect algebra.

Let P and Q be S-sets. A map $\phi: P \to Q$ is a *morphism* if $\phi(0) = 0$, $\phi(1) = 1$, and if $(a, b) \in D(\oplus)$, then $(\phi(a), \phi(b)) \in D(\oplus)$ and $\phi(a) \oplus \phi(b) = \phi(a \oplus b)$. If ϕ is a bijective morphism and ϕ^{-1} is a morphism, then ϕ is an *isomorphism*.

Let P, Q, and R be S-sets. A map $\beta: P \times Q \to R$ is called a *bimorphism* if the following conditions hold:

(1) $\beta(1, 1) = 1$ and $\beta(0, b) = \beta(a, 0) = 0$ for all $a \in P$, $b \in Q$.

(2) If $(a, b) \in D(\oplus)$, then $(\beta(a, c), \beta(b, c)) \in D(\oplus)$ for all $c \in Q$ and

$$\beta(a, c) \oplus \beta(b, c) = \beta(a \oplus b, c)$$

(3) If $(c, d) \in D(\oplus)$, then $(\beta(a, c), \beta(a, d)) \in D(\oplus)$ for all $a \in P$ and

$$\beta(a, c) \oplus \beta(a, d) = \beta(a, c \oplus d)$$

It is interesting to note that a bimorphism $\beta: P \times Q \to R$ always satisfies the following apparently stronger conditions:

(4) If $(a, b) \in D(\oplus)$, then $(\beta(a, c), \beta(b, d)) \in D(\oplus)$ for all $c, d \in Q$.

(5) If $(c, d) \in D(\oplus)$, then $(\beta(a, c), \beta(b, d)) \in D(\oplus)$ for all $a, b \in P$.

For example, to prove (4), suppose a, $b \in P$ and c, $d \in Q$ with $(a, b) \in D(\oplus)$. Letting $c' \in S(c)$, $d' \in S(d)$, we have

$$(\beta(a, c') \oplus \beta(a, c), \beta(b, d \oplus d')) = (\beta(a, 1), \beta(b, 1)) \in D(\oplus)$$

Hence,

$$(\beta(a, c), \beta(b, d) \oplus \beta(b, d')) = (\beta(a, c), \beta(b, d \oplus d')) \in D(\oplus)$$

so $(\beta(a, c), \beta(b, d)) \in D(\oplus)$. The proof of (5) is similar.

Due to the associativity of $\oplus$, we can omit parentheses when writing the sum of three or more elements of an S-set when the sum exists. It easily follows that

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{2.1}$$

for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ when the sum on the left-hand side exists. Moreover,

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{2.2}$$

when the sum on the left-hand side exists.

Let P, Q, and T be S-sets and let $\tau: P \times Q \to T$ be a bimorphism. We say that (T, τ) is a *tensor product* of P and Q if the following conditions are satisfied.

(1) If R is an S-set and $\beta: P \times Q \to R$ is a bimorphism, then there exists a morphism $\phi: T \to R$ such that $\beta = \phi \circ \tau$.

(2) Every element of T is a finite sum of elements of the form $\tau(a, b)$ with $a \in P$, $b \in Q$.

If P, Q are affect algebras, the definition of a tensor product of P and Q is similar. The only change is that T and R are assumed to be effect algebras. The next lemma shows that if a tensor product exists, it is unique.

Lemma 2.1. If (T, τ) and (T^*, τ^*) are tensor products of P and Q, then there exists a unique isomorphism $\phi: T \to T^*$ such that $\phi(\tau(a, b)) = \tau^*(a, b)$ for all $a \in P$, $b \in Q$.

Proof. By definition of tensor products, there exist morphisms $\phi: T \to T^*$ and $\phi^*: T^* \to T$ such that $\tau^* = \phi \circ \tau$ and $\tau = \phi^* \circ \tau^*$. If $t \in T$, then t has the form

$$t = \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$$

Hence,

$$\begin{aligned}\phi^* \circ \phi(t) &= \phi^*[\phi \circ \tau(a_1, b_1) \oplus \cdots \oplus \phi \circ \tau(a_n, b_n)] \\ &= \phi^* \circ \tau^*(a_1, b_1) \oplus \cdots \oplus \phi^* \circ \tau^*(a_n, b_n) \\ &= \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n) = t\end{aligned}$$

Similarly, $\phi \circ \phi^*(t^*) = t^*$ for every $t^* \in T^*$. Thus, ϕ is an isomorphism and applying Condition 2, we see that ϕ is unique. ■

3. TENSOR PRODUCTS OF S-SETS

In this section we show that the tensor product (T, τ) of any two S-sets P and Q exists. Moreover, we give a concrete representation of (T, τ).

Theorem 3.1. If P and Q are S-sets, then their tensor product (T, τ) exists.

Proof. Let $\mathcal{F}(P \times Q)$ be the set of all finite formal sums $a_1b_1 + \cdots + a_nb_n$, $a_i \in P$, $b_i \in Q$, $a_i \neq 0$, $b_i \neq 0$, $i = 1, \ldots, n$. We identify two formal sums if they coincide except possibly for the order of their terms. A *subsum* of $a_1b_1 + \cdots + a_nb_n$ is any $A \in \mathcal{F}(P \times Q)$ of the form

$$A = a_{i_1}b_{i_1} + \cdots + a_{i_m}b_{i_m}$$

where $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$. If $A = a_1b_1 + \cdots + a_nb_n \in \mathcal{F}(P \times Q)$ and $B = c_1d_1 + \cdots + c_md_m \in \mathcal{F}(P \times Q)$, we define $A + B \in \mathcal{F}(P \times Q)$ by

$$A + B = a_1b_1 + \cdots + a_nb_n + c_1d_1 + \cdots c_md_m$$

with the convention that $A + \emptyset = A$, where $\emptyset \in \mathcal{F}(P \times Q)$ is the empty sum. For $A \in \mathcal{F}(P \times Q)$ we define the following *operations*:

(1) If $ac + bc$ is a subsum of A with $(a, b) \in D(\oplus)$, replace $ac + bc$ by $(a \oplus b)c$. If $(a \oplus b)c$ is a subsum of A with $a, b \neq 0$, replace $(a \oplus b)c$ by $ac + bc$.

(2) If $ac + ad$ is a subsum of A with $(c, d) \in D(\oplus)$, replace $ac + ad$ by $a(c \oplus d)$. If $a(c \oplus d)$ is a subsum of A with $c, d \neq 0$, replace $a(c \oplus d)$ by $ac + ad$.

For $A, B \in \mathcal{F}(P \times Q)$ if A can be transformed into B by a finite sequence of operations, we write $A \sim B$. It is clear that $\sim$ is an equivalence relation. We now observe that $A \sim A_1$ and $B \sim B_1$ imply that $A + B \sim A_1 + B_1$. Indeed, on $A + B$ we first apply the sequence of operations that takes A into A_1 and then apply the sequence of operations that takes B into B_1.

Let $\mathcal{T}(P \times Q) = \{A \in \mathcal{F}(P \times Q): A \sim 11\}$. Notice that for every $a \in P$, $b \in Q$, $a, b \neq 0$, there exists an $A \in \mathcal{T}(P \times Q)$ such that ab is a term of A. Indeed, let $a' \in S(a)$, $b' \in S(b)$ and let $A = ab + a'b + 1b'$. Then

$$A \sim (a \oplus a')b + 1b' = 1b + 1b' \sim 1(b \oplus b') = 11$$

so $A \in \mathcal{T}(P \times Q)$. Denote the set of all subsums of elements of $\mathcal{T}(P \times Q)$ by $\mathcal{E}(P \times Q)$. For $A \in \mathcal{E}(P \times Q)$ we define the equivalence class

$$\Pi(A) = \{B \in \mathcal{F}(P \times Q): B \sim A\}$$

Notice that if $B \in \Pi(A)$, then $B \in \mathscr{E}(P \times Q)$. Indeed, there exists a $C \in \mathscr{F}(P \times Q)$ such that $A + C \in \mathscr{T}(P \times Q)$. Hence, $B + C \sim A + C \sim 11$, so $B + C \in \mathscr{T}(P \times Q)$. Hence, $B \in \mathscr{E}(P \times Q)$. Also, if $A \in \mathscr{E}(P \times Q)$ and if A_1 is a subsum of A, then $A_1 \in \mathscr{E}(P \times Q)$.

Using the notation $\Pi(11) = 1$, we have $\Pi(E) = 1$ for every $E \in \mathscr{T}(P \times Q)$. Using the notation $\Pi(\varnothing) = 0$, we have $\{\varnothing\} = 0$. Letting

$$T = \Pi(P \times Q) = \{\Pi(A): A \in \mathscr{E}(P \times Q)\}$$

we have $0, 1 \in T$. We define a partial binary operation $\oplus: D(\oplus) \to T$, $D(\oplus) \subseteq T \times T$, as follows:

$$D(\oplus) = \{(\Pi(A), \Pi(B)): A + B \in \mathscr{E}(P \times Q)\}$$

and if $(\Pi(A), \Pi(B)) \in D(\oplus)$, then $\Pi(A) \oplus \Pi(B) = \Pi(A + B)$. To show that $\oplus$ is well defined, suppose $A_1, B_1 \in \mathscr{E}(P \times Q)$ and $A_1 \sim A$, $B_1 \sim B$. Then $A_1 + B_1 \sim A + B$. Hence, $A_1 + B_1 \in \mathscr{E}(P \times Q)$ if and only if $A + B \in \mathscr{E}(P \times Q)$ and in this case $\Pi(A_1 + B_1) = \Pi(A + B)$.

We now show that $(T, 0, 1, \oplus)$ is an S-set. First, $0 \neq 1$, since otherwise $\varnothing \sim 11$ and this is clearly impossible. It is clear that $\oplus$ is commutative. Suppose that

$$(\Pi(A), \Pi(B)), (\Pi(A) \oplus \Pi(B), \Pi(C)) \in D(\oplus)$$

Then $A + B + C \in \mathscr{E}(P \times Q)$ and $(\Pi(A) \oplus \Pi(B)) \oplus \Pi(C) = \Pi(A + B + C)$. Hence, $B + C \in \mathscr{E}(P \times Q)$ and $\Pi(B) \oplus \Pi(C) = \Pi(B + C)$. Again, $(\Pi(A), \Pi(B) \oplus \Pi(C)) \in D(\oplus)$ and $\Pi(A) \oplus (\Pi(B) \oplus \Pi(C)) = \Pi(A + B + C)$. If $A \in \mathscr{E}(P \times Q)$, then there exists a $B \in \mathscr{E}(P \times Q)$ such that $A + B \sim 11$. Hence, $(\Pi(A), \Pi(B)) \in D(\oplus)$ and

$$\Pi(A) \oplus \Pi(B) = \Pi(A + B) = \Pi(11) = 1$$

so Condition (S3) holds. Since $\varnothing + A = A$, we have $(0, \Pi(A)) \in D(\oplus)$ for every $\Pi(A) \in T$ and $0 + \Pi(A) = \Pi(\varnothing) + \Pi(A) = \Pi(A)$, so Condition (S4) holds. Finally, to show that S5 holds, suppose $(\Pi(A), \Pi(B)) \in D(\oplus)$ and $\Pi(A) \oplus \Pi(B) = 0$. Then $\Pi(A + B) = 0$, so $A + B = \varnothing$. Hence, $A = B = \varnothing = 0$.

Define the map $\tau: P \times Q \to T$ by $\tau(a, b) = \Pi(ab)$ whenever $a, b \neq 0$ and $\tau(a, b) = 0$ otherwise. This is consistent because we have shown that $ab \in \mathscr{E}(P \times Q)$ whenever $a, b \neq 0$. Then $\tau(1, 1) = \Pi(11) = 1$ and $\tau(0, b) = \tau(a, 0) = 0$ for every $a \in P$, $b \in Q$. Let $a, b \in P$, $c \in Q$ with $a, b, c \neq 0$ and $(a, b) \in D(\oplus)$. Then for $c' \in S(c)$ and $(a \oplus b)' \in S(a \oplus b)$ we have

$$\begin{aligned} &ac + bc + ac' + bc' + (a \oplus b)'1 \\ &\quad \sim a(c \oplus c') + b(c \oplus c') + (a \oplus b)'1 \\ &\quad = a1 + b1 + (a \oplus b)' \sim (a \oplus b)1 + (a \oplus b)'1 \sim 11 \end{aligned}$$

Hence, $ac + bc \in \mathscr{E}(P \times Q)$, so $(\Pi(ac), \Pi(bc)) \in D(\oplus)$ and

$$\Pi(ac) \oplus \Pi(bc) = \Pi(ac + bc) = \Pi((a \oplus b)c)$$

Therefore, $(\tau(a, c), \tau(b, c)) \in D(\oplus)$ and $\tau(a, c) \oplus \tau(b, c) = \tau(a \oplus b, c)$. If a, b, or c is 0, we proceed in a similar way. Similarly, if $a \in P$ and $b, c \in Q$ with $(b, c) \in D(\oplus)$, then $(\tau(a, b), \tau(a, c)) \in D(\oplus)$ and $\tau(a, b) \oplus \tau(a, c) = \tau(a, b \oplus c)$. We conclude that τ is a bimorphism. To show that $\tau(P \times Q)$ generates T, let $\Pi(A) \in T$, where $A = a_1b_1 + \cdots + a_nb_n$. Then

$$\Pi(A) = \Pi(a_1b_1) \oplus \cdots \oplus \Pi(a_nb_n) = \tau(a_1, b_1) \oplus \cdots \oplus \tau(a_n, b_n)$$

Finally, to show that (T, τ) is the tensor product of P and Q, let R be an S-set and let β: $P \times Q \to R$ be a bimorphism. Let $A, B \in \mathscr{F}(P \times Q)$, where $A = a_1b_1 + \cdots + a_nb_n$, $B = c_1d_1 + \cdots + c_md_m$, and suppose that $A \sim B$. We shall show that if $\beta(a_1, b_1) \oplus \cdots \oplus \beta(a_n, b_n)$ exists, then $\beta(c_1, d_1) \oplus \cdots \oplus \beta(c_m, d_m)$ exists and the two expressions coincide. Indeed, if we can transform A into B by a single operation, then B has the form

$$(a_1 \oplus a_2)b_1 + a_3b_3 + \cdots + a_nb_n$$

where $b_2 = b_1$, or the form

$$cb_1 + db_1 + a_2b_2 + \cdots + a_nb_n$$

where $c \oplus d = a_1$. In the first case, we have

$$\begin{aligned}&\beta(a_1 \oplus a_2, b_1) \oplus \beta(a_3, b_3) \oplus \cdots \oplus \beta(a_n, b_n)\\ &\quad = \beta(a_1, b_1) \oplus \beta(a_2, b_2) \oplus \cdots \oplus \beta(a_n, b_n)\end{aligned}$$

and in the second case, we have

$$\begin{aligned}&\beta(c, b_1) \oplus \beta(d, b_1) \oplus \beta(a_2, b_2) \oplus \cdots \oplus \beta(a_n, b_n)\\ &\quad = \beta(c \oplus d, b_1) \oplus \beta(a_2, b_2) \oplus \cdots \oplus \beta(a_n, b_n)\\ &\quad = \beta(a_1, b_1) \oplus \beta(a_2, b_2) \oplus \cdots \oplus \beta(a_n, b_n)\end{aligned}$$

The result now follows by induction on the number of operations. Applying this result, we conclude that for any $a_1b_1 + \cdots + a_nb_n \in \mathscr{T}(P \times Q)$ we have

$$\beta(a_1, b_1) \oplus \cdots \oplus \beta(a_n, b_n) = 1$$

It follows from (2.2) that if $A = c_1d_1 + \cdots + c_md_m \in \mathscr{E}(P \times Q)$, then $\beta(c_1, d_1) \oplus \cdots \oplus \beta(c_m, d_m)$ is defined and we define ϕ: $T \to R$ by

$$\phi(\Pi(A)) = \beta(c_1, d_1) \oplus \cdots \oplus \beta(c_m, d_m)$$

if $A \neq \varnothing$ and otherwise $\phi(0) = 0$. Our previous result shows that ϕ is well defined. To show that ϕ: $T \to R$ is a morphism, we have $\phi(1) = \beta(1, 1) = 1$.

Suppose that $\Pi(A), \Pi(B) \in T$, and $(\Pi(A), \Pi(B)) \in D(\oplus)$. Letting $A = a_1b_1 + \cdots + a_nb_n$ and $B = c_1d_1 + \cdots + c_md_m$, we have that $A + B \in \mathscr{E}(P \times Q)$ and applying (2.2) gives

$$\begin{aligned}\phi(\Pi(A) \oplus \Pi(B)) &= \phi(\Pi(A + B)) \\ &= \beta(a_1, b_1) \oplus \cdots \oplus \beta(a_n, b_n) \oplus \beta(c_1, d_1) \\ &\quad \oplus \cdots \oplus \beta(c_m, d_m) \\ &= \phi(\Pi(A)) \oplus \phi(\Pi(B))\end{aligned}$$

Finally, to show that $\beta = \phi \circ \tau$, we have for $a, b \neq 0$ that

$$\phi[\tau(a, b)] = \phi[\Pi(ab)] = \beta(a, b)$$

and if a or b is 0, then $\phi[\tau(a, b)] = 0 = \beta(a, b)$. ■

Let P and Q be S-sets and let T be their tensor product space as constructed in Theorem 3.1. Using the notation in the proof of Theorem 3.1, let $A, B \in \mathscr{T}(P \times Q)$. The maximal (in length) common subsum of A and B is denoted $A \cap B$. (Of course, $A \cap B$ could be $\varnothing$.) We then have $A = A_1 + A \cap B$ and $B = B_1 + A \cap B$ for some $A_1, B_1 \in \mathscr{E}(P \times Q)$. We then use the notation $A_1 = A - A \cap B$, $B_1 = B - A \cap B$. The next corollary characterizes those T that are effect algebras. Moreover, if P and Q are effect algebras, it gives a sufficient condition for the existence of their effect algebra tensor product. This condition may be useful for establishing the existence of effect algebra tensor products for particular examples.

Corollary 3.2. (1) T is an effect algebra if and only if $A - A \cap B \sim B - A \cap B$ for every $A, B \in \mathscr{T}(P \times Q)$. (2) If P and Q are effect algebras and if $A - A \cap B \sim B - A \cap B$ for every $A, B \in \mathscr{T}(P \times Q)$, then their effect algebra tensor product exists and equals (T, τ).

Proof. (1) Suppose $A - A \cap B \sim B - A \cap B$ for every $A, B \in \mathscr{T}(P \times Q)$. If $A + C, B + C \in \mathscr{T}(P \times Q)$, since C is a subsum of $(A + C) \cap (B + C)$, it follows that $A \sim B$. Now suppose $A, B, C \in \mathscr{E}(P \times Q)$ and

$$\Pi(A) \oplus \Pi(B) = 1 = \Pi(A) \oplus \Pi(C)$$

Then $A + B, A + C \in \mathscr{T}(P \times Q)$, so by the above, $B \sim C$. Hence, $\Pi(B) = \Pi(C)$, so (S7) holds for T. Next, suppose $(\Pi(A), 1) \in D(\oplus)$. Then $A + 11 \in \mathscr{E}(P \times Q)$, so there exists a $B \in \mathscr{E}(P \times Q)$ such that $B + A + 11 \in \mathscr{T}(P \times Q)$. Since $\varnothing + 11 \in \mathscr{T}(P \times Q)$, we have $B + A \sim \varnothing$. Hence, $B = A = \varnothing$, so $\Pi(A) = 0$ and (S6) holds for T. Conversely, suppose T is an effect algebra and $A, B \in \mathscr{T}(P \times Q)$. Then

$$\Pi(A - A \cap B) \oplus \Pi(A \cap B) = 1 = \Pi(B - A \cap B) \oplus \Pi(A \cap B)$$

Applying the cancellation law (Foulis and Bennett, 1994), we have $\Pi(A - A \cap B) = \Pi(B - A \cap B)$, so $A - A \cap B \sim B - A \cap B$.

(2) If the condition holds, then T is an effect algebra, so (T, τ) is the effect algebra tensor product of P and Q. ■

4. S-SETS AND EFFECT ALGEBRAS

In this section we use morphisms to study relationships between S-sets and effect algebras. We also employ S-set tensor products to obtain information about effect algebra tensor products. Our first result shows that an S-set is an effect algebra if and only if it admits an injective morphism into an effect algebra.

Lemma 4.1. If P is an S-set, R is an effect algebra, and there exists an injective morphism $\phi: P \to R$, then P is an effect algebra.

Proof. For $a \in P$ there exists a $b \in P$ such that $(a, b) \in D(\oplus)$ and $a \oplus b = 1$. Suppose $(a, c) \in D(\oplus)$ and $a \oplus c = 1$. Then $(\phi(a), \phi(b))$, $(\phi(a), \phi(c)) \in D(\oplus)$ and

$$\phi(a) \oplus \phi(b) = 1 = \phi(a) \oplus \phi(c)$$

Applying the cancellation law (Foulis and Bennett, 1994), we conclude that $\phi(c) = \phi(b)$. Since ϕ is injective, $c = b$, so $|S(a)| = 1$. Suppose $a \in P$ with $(a, 1) \in D(\oplus)$. Then $(\phi(a), 1) \in D(\oplus)$, so $\phi(a) = 0$. Since ϕ is injective, $a = 0$. ■

A subset P_1 of an S-set P is a *sub-S-set* of P if P_1 satisfies the following conditions:

(1) $0, 1 \in P_1$.
(2) If $a, b \in P_1$ with $(a, b) \in D(\oplus)$, then $a \oplus b \in P_1$.
(3) If $a \in P_1$, then there is a $b \in P_1$ such that $(a, b) \in D(\oplus)$ and $a \oplus b = 1$.

If P is an effect algebra and $P_1 \subseteq P$, then P_1 is a *sub-effect algebra* of P if P_1 is a sub-S-set of P. In this case (3) can be replaced by the simpler condition: if $a \in P_1$, then $a' \in P_1$. It is easy to verify that a sub-S-set P_1 with $\oplus$ restricted to P_1 is an S-set. The same holds for a sub-effect algebra. If P and Q are S-sets, then a morphism $\phi: P \to Q$ is *strong* if $(\phi(a), \phi(b)) \in D(\oplus)$ implies that there is a $c \in P$ such that $\phi(c) = \phi(a) \oplus \phi(b)$.

Lemma 4.2. If P and Q are S-sets, then a morphism $\phi: P \to Q$ is strong if and only if $\phi(P)$ is a sub-S-set of Q.

Proof. Suppose $\phi(P)$ is a sub-S-set of Q. If $(\phi(a), \phi(b)) \in D(\oplus)$, then $\phi(a) \oplus \phi(b) \in \phi(P)$. Hence, there is a $c \in P$ such that $\phi(c) = \phi(a) \oplus \phi(b)$, so ϕ is strong. Conversely, suppose that $\phi\colon P \to Q$ is strong. Then $0 = \phi(0) \in \phi(P)$ and $1 = \phi(1) \in \phi(P)$. If $(\phi(a), \phi(b)) \in D(\oplus)$, then there is a $c \in P$ such that $\phi(a) \oplus \phi(b) = \phi(c) \in \phi(P)$. Finally, if $\phi(a) \in \phi(P)$, then there is a $b \in P$ such that $(a, b) \in D(\oplus)$ and $a \oplus b = 1$. Hence, $(\phi(a), \phi(b)) \in D(\oplus)$ and $\phi(a) \oplus \phi(b) = 1$. ■

We have seen in Lemma 4.1 that if an S-set P has an injective morphic image in an effect algebra, then P is an effect algebra. If such an image does not exist, does P still "contain" an effect algebra? More precisely, does there exist an equivalence relation $\approx$ on P such that $P/\approx$ can be organized into an effect algebra and the canonical map $\psi(a) = [a]_{\approx}$ is a morphism? A necessary condition for this to hold is that there exists a surjective morphism $\phi\colon P \to R$ where R is an effect algebra. We now show that this condition is sufficient.

To begin, let P and Q be S-sets and let $\phi\colon P \to Q$ be a surjective morphism. For $a, b \in P$, we write $a \approx b$ if $\phi(a) = \phi(b)$. It is clear that $\approx$ is an equivalence relation and we denote the equivalence class containing a by $[a]$. Using the notation

$$P/\phi = \{[a]: a \in P\}$$

we define the canonical surjective $\psi\colon P \to P/\phi$ by $\psi(a) = [a]$ and the map $\overline{\phi}\colon P/\phi \to Q$ by $\overline{\phi}([a]) = \phi(a)$. Notice that $\overline{\phi}$ is well defined. We define the partial binary operation $\overline{\oplus}$ on P/ϕ as follows. Declare that $([a], [b]) \in D(\overline{\oplus})$ if $(\phi(a), \phi(b)) \in D(\oplus)$ and in this case $[a] \overline{\oplus} [b] = [c]$, where $c \in P$ satisfies $\phi(c) = \phi(a) \oplus \phi(b)$. Again we see that $\overline{\oplus}$ is well defined.

Theorem 4.3. If P and Q are S-sets and $\phi\colon P \to Q$ is a surjective morphism, then $\mathcal{P} = (P/\phi, [0], [1], \overline{\oplus})$ is an S-set, $\psi\colon P \to P/\phi$ is a surjective morphism, $\overline{\phi}\colon P/\phi \to Q$ is an isomorphism, and $\phi = \overline{\phi} \circ \psi$.

Proof. It is clear that $[0] \neq [1]$ and that S1 holds for $\mathcal{P}$. To show that $\mathcal{P}$ satisfies S2, suppose $([a], [b]), ([a] \overline{\oplus} [b], [c]) \in D(\overline{\oplus})$. Then $(\phi(a), \phi(b)) \in D(\oplus)$ and $[a] \overline{\oplus} [b] = [d]$, where $\phi(d) = \phi(a) \oplus \phi(b)$. Moreover, $(\phi(d), \phi(c)) \in D(\oplus)$ and

$$([a] \overline{\oplus} [b]) \overline{\oplus} [c] = [d] \overline{\oplus} [c] = [e]$$

where $\phi(e) = \phi(d) \oplus \phi(c)$. Then $(\phi(b), \phi(c)), (\phi(a), \phi(b) \oplus \phi(c)) \in D(\oplus)$ and

$$\phi(a) \oplus (\phi(b) \oplus \phi(c)) = (\phi(a) \oplus \phi(b)) \oplus \phi(c) = \phi(e)$$

Letting $\phi(f) = \phi(b) \oplus \phi(c)$, we have $([b], [c]) \in D(\overline{\oplus})$ and $[b] \oplus [c] = [f]$. Moreover, $([a], [b] \oplus [c]) \in D(\overline{\oplus})$ and

$$[a] \oplus ([b] \overline{\oplus} [c]) = [e]$$

To verify S3 for $\mathcal{P}$, let $[a] \in P/\phi$. Then there exists a $b \in P$ such that $(a, b) \in D(\oplus)$ and $a \oplus b = 1$. Then $(\phi(a), \phi(b)) \in D(\oplus)$ and $\phi(a) \oplus \phi(b) = \phi(1) = 1$. Hence, $([a], [b]) \in D(\overline{\oplus})$ and $[a] \overline{\oplus} [b] = [1]$. To show that $\mathcal{P}$ satisfies S4, it is clear that $([0], [a]) \in D(\overline{\oplus})$ and $[0] \overline{\oplus} [a] = [a]$ for all $a \in P$. Finally, to show that $\mathcal{P}$ satisfies S5, suppose that $([a], [b]) \in D(\overline{\oplus})$ and $[a] \overline{\oplus} [b] = [0]$. Then $(\phi(a), \phi(b)) \in D(\oplus)$ and $\phi(a) \oplus \phi(b) = 0$. Hence, $\phi(a) = \phi(b) = 0$, so $[a] = [b] = [0]$. We conclude that $\mathcal{P}$ is an S-set.

To verify that ψ is a morphism, suppose $(a, b) \in D(\oplus)$. Then $(\phi(a), \phi(b)) \in D(\oplus)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. Hence, $([a], [b]) \in D(\overline{\oplus})$ and

$$\psi(a \oplus b) = [a \oplus b] = [a] \overline{\oplus} [b] = \psi(a) \oplus \psi(b)$$

To show that $\overline{\phi}$ is a morphism, suppose $([a], [b]) \in D(\overline{\oplus})$. Then $(\phi(a), \phi(b)) \in D(\oplus)$ and $[a] \overline{\oplus} [b] = [c]$, where $\phi(c) = \phi(a) \oplus \phi(b)$. Hence,

$$\overline{\phi}([a] \overline{\oplus} [b]) = \overline{\phi}([c]) = \phi(c) = \phi(a) \oplus \phi(b) = \overline{\phi}([a]) \overline{\oplus} \overline{\phi}([b])$$

To show that $\overline{\phi}$ is injective, we have $\overline{\phi}([a]) = \overline{\phi}([b])$ implies $\phi(a) = \phi(b)$, so $[a] = [b]$. Finally, if $(\overline{\phi}([a]), \overline{\phi}([b])) \in D(\oplus)$, then $(\phi(a), \phi(b)) \in D(\oplus)$, so $([a], [b]) \in D(\overline{\oplus})$ and $[a] \overline{\oplus} [b] = [c]$, where $\phi(c) = \phi(a) \oplus \phi(b)$. Hence,

$$\begin{aligned}\overline{\phi}^{-1}(\overline{\phi}([a]) \oplus \overline{\phi}([b])) &= \overline{\phi}^{-1}(\overline{\phi}([c])) = [c] = [a] \overline{\oplus} [b] \\ &= \overline{\phi}^{-1}(\overline{\phi}([a])) \overline{\oplus} \overline{\phi}^{-1}(\overline{\phi}([b]))\end{aligned}$$

Thus, $\overline{\phi}$ is an isomorphism and it is clear that $\phi = \overline{\phi} \circ \psi$. ■

Notice that a similar theorem holds if $\phi: P \to Q$ is a strong morphism. The only difference is that now $\overline{\phi}: P/\phi \to \phi(P)$ is an isomorphism.

Corollary 4.4. Let P be an S-set, R an effect algebra, and $\phi: P \to R$ a surjective morphism. Then $\mathcal{P} = (P/\phi, [0], [1], \overline{\oplus})$ is an effect algebra, $\psi: P \to P/\phi$ is a surjective morphism, $\overline{\phi}: P/\phi \to R$ is an isomorphism, and $\phi = \overline{\phi} \circ \psi$.

Proof. It follows from Theorem 4.3 that $\mathcal{P}$ is an S-set and it suffices to show that $\mathcal{P}$ is an effect algebra. To verify S6 for $\mathcal{P}$, suppose that $([a], [1]) \in D(\overline{\oplus})$. Then $(\phi(a), 1) \in D(\oplus)$ and since R is an effect algebra, $\phi(a) = 0$. Hence, $[a] = [0]$. To verify S7 for $\mathcal{P}$, let $[a] \in P/\phi$. Since $\mathcal{P}$ is an S-set, there exists a $[b] \in P/\phi$ such that $([a], [b]) \in D(\overline{\oplus})$ and $[a] \overline{\oplus} [b] = [1]$. Suppose $([a], [c]) \in D(\overline{\oplus})$ and $[a] \overline{\oplus} [c] = [1]$. Then

$$\phi(a) \oplus \phi(b) = 1 = \phi(a) \oplus \phi(c)$$

Since R is an effect algebra, the cancellation law (Foulis and Bennett, 1994) gives $\phi(c) = \phi(a)$. Hence, $[c] = [b]$, so $|S([a])| = 1$. ■

If P and Q are effect algebras, it is known that their effect algebra tensor product need not exist (Gudder and Greechie, 1996). We now characterize the existence of their effect algebra tensor product in terms of their S-set tensor product, which always exists.

Lemma 4.5. Let P and Q be effect algebras and let (T, τ) be their S-set tensor product. Then their effect algebra tensor product exists if and only if there exists a morphism $\phi: T \to R$ where R is an effect algebra.

Proof. Suppose the effect algebra tensor product $(\hat{T}, \tilde{\tau})$ of P and Q exists. Then by definition of (T, τ), there exists a morphism $\phi: T \to \hat{T}$. Conversely, suppose there exists a morphism $\phi: T \to R$, where R is an effect algebra. Then $\phi \circ \tau$ is a bimorphism from $P \times Q$ into R. It follows from Dvurečenskij (n.d.) that the effect algebra tensor product of P and Q exists. ■

Let P and Q be effect algebras and let (T, τ) be their S-set tensor product. Suppose that their effect algebra tensor product $(\hat{T}, \hat{\tau})$ exists. Then there exists a unique morphism $\phi: T \to \hat{T}$ such that $\hat{\tau} = \phi \circ \tau$. In this way, we can obtain $(\hat{T}, \hat{\tau})$ from (T, τ). The next result gives more information if ϕ is strong.

Lemma 4.6. If $\phi: T \to \hat{T}$ is a strong morphism, then $(T/\phi, [0], [1], \overline{\oplus})$ is an effect algebra and $(T/\phi, \psi \circ \tau)$ is the effect algebra tensor product of P and Q.

Proof. It follows from Lemma 4.2 and Corollary 4.4 that $(T/\phi, [0], [1], \overline{\oplus})$ is an effect algebra and that $\overline{\phi}: T/\phi \to \phi(T) \subseteq \hat{T}$ is an isomorphism. It thus suffices to prove that $\phi(T) = \hat{T}$. Let $s \in \hat{T}$. Since $\hat{\tau}(P \times Q)$ generates $\hat{T}$, there exist $a_i \in P$, $b_i \in Q$, $i = 1, \ldots, n$, such that

$$s = \hat{\tau}(a_1, b_1) \oplus \cdots \oplus \hat{\tau}(a_n, b_n) = \phi \circ \tau(a_1, b_1) \oplus \cdots \oplus \phi \circ \tau(a_n, b_n)$$

But since ϕ is strong, there exists a $t \in T$ such that $\phi(t) = s$. ■

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